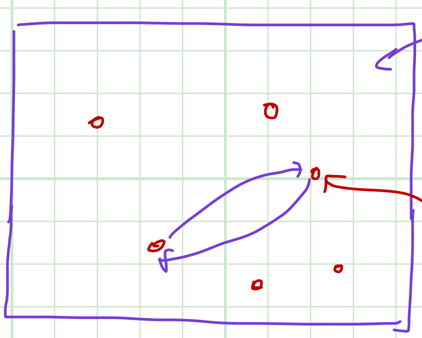
Topologically protected quantum computation + non-abelien statistics.

2D equivalent of the Kitaev wire is a p-wave superconductor.
In this case the Majorana zero modes "live" inside half-flux vortices.



2D p-wave superconductor

½ flux vortex (with one Majorana zero mode)

What is the number of ground states?

If we have n [½ vortices]

$\Rightarrow$ n Majorana zero modes

$\Rightarrow$ $n/2$ complex fermions with zero energy

$\Rightarrow$ $2^{n/2}$ ground states. $\Rightarrow$ This is the Hilbert space on which we can do quantum computation.

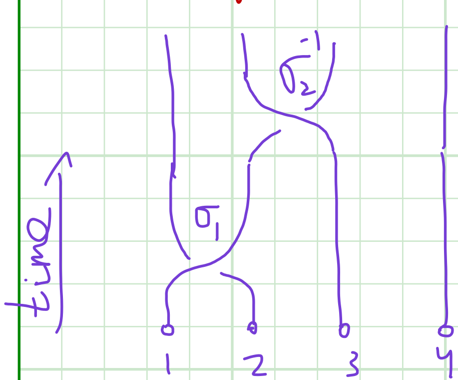
Braiding: we can go between different ground states by adiabatically (slowly swapping location of vortices).

$\Rightarrow$ This is a consequence of non-abelian statistics

abelian $\Leftrightarrow$ swapping position of excitations has no effect on the state [up to a phase]

non-abelian $\Leftrightarrow$ swapping position of excitations changes the wave function

Braid group:



The braid group describes "braiding" operations on particles $\Rightarrow$ action of elements of this group on NAA's describes topo-protected quantum computation.

$\Rightarrow$ The group also describes how to braid a bunch of ropes/strings.

generators of the braid group

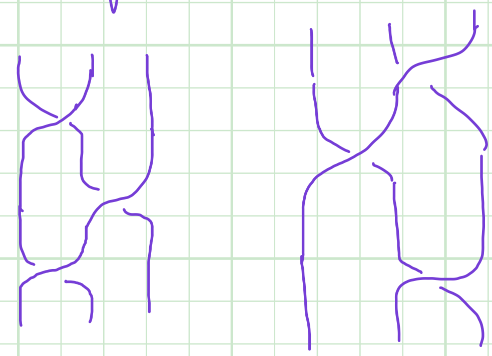
$\Rightarrow$ σ_i swaps particle i and $i+1$ with with particle i going "over the top"

$\Rightarrow$ σ_i^{-1} = inverse operation of σ_i, swap with i going "under" $i+1$.

properties of generators

$\Rightarrow$ $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| \geq 2$ $\Rightarrow$ why

$\Rightarrow$ $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$



$\Rightarrow$ These rules let us compare braids.

Anyons :

consider swapping two particles

$$\psi(r_1, r_2) \to \psi(r_2, r_1)$$

what is the relation between the two wave functions?

Bosons: $\psi(r_1, r_2) = \psi(r_2, r_1)$ $\Rightarrow$ $\theta = 0$

Fermions: $\psi(r_1, r_2) = -\psi(r_2, r_1)$ $\Rightarrow$ $\theta = \pi$

Anyons: $\psi(r_1, r_2) = e^{i\theta} \psi(r_2, r_1)$ $(**)$

$\Rightarrow$ The first two examples are special cases of the last, as noted.

$\Rightarrow$ Low energy excitations of 2D systems made up of Fermions and Bosons can be Anyons.

$\Rightarrow$ The anyons defined by $(**)$ are Abelian
The order in which we exchange particles does <u>not</u> matter

e.g. consider the action of $\sigma_1 \sigma_2$ and $\sigma_2 \sigma_1$

$$\psi(r_1, r_2, r_3) \to e^{i\theta} \psi(r_1, r_2, r_3) \to e^{2i\theta} \psi(r_1, r_2, r_3)$$

1 2

2 1

Non-abelian anyons

We can further generalize the expression for swapping particles by replacing the phase θ with a matrix and ψ with a vector.

In this generalization the order of operations becomes important: the action of $\sigma_1 \sigma_2$ and $\sigma_2 \sigma_1$ on the wave function is different!

$$M_{12} M_{23} \neq M_{23} M_{12}$$

[if M's do not commute]
explicit example soon

Quantum operations and the Braid group

⇒ As we have argued the ground state of a system of Non-abelian anyons is degenerate

⇒ Ground state wave function must be a vector that describes the superposition of the degenerate states.

⇒ The braid group induces unitary transformations on the ground state wave function

$$\psi \to U[\sigma_i]\psi$$

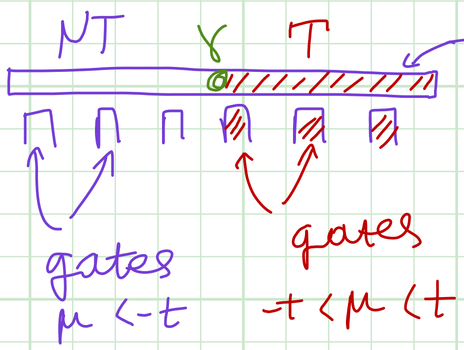
⇒ as we have argued the U's do not have to commute with each other.

Observation: We can use braiding to perform quantum computation

⇒ Small problem: Majorana zero modes too simple to implement generic quantum computations with braiding alone, must use more complicated anyons or supplements with non-topologically protected quantum operations.

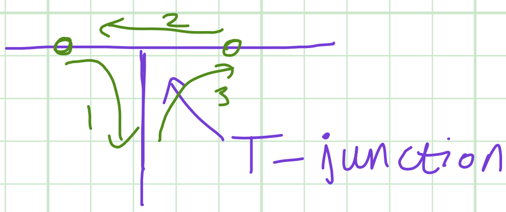
How to braid wires?

(1) we want to be able to move Majorana zero modes since μ can be used to tune between T + NT, let's buid a system in which we can adjust μ:



$\Rightarrow$ changing charge on gates lets us move the domain wall between T and NT and hence move the MZM.

(2) We need to step out of 1D slightly so that we can move MZM's around each other. use T-junction.



see Alicea et al. Nature Physics 7, 412 ('11)

How to make topological superconducting wires?

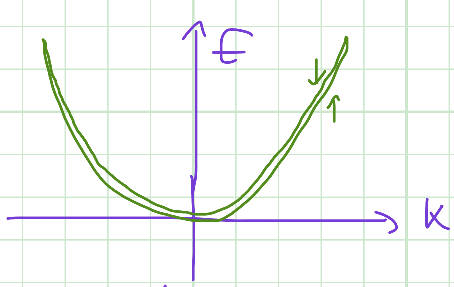
Prerequisites:

(1) Single species of fermion [single spin]

(2) Superconductivity

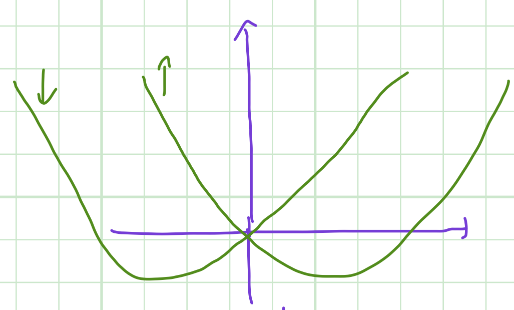
Start with (1): Make single species using combination of spin-orbit + Zeeman in a 1D wire.

Dispersion:



plain wire

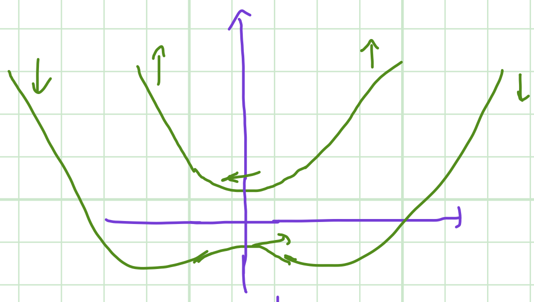
$$H = \sum_{k\sigma} \frac{k^2}{2m} c_{k\sigma}^{+} c_{k\sigma}$$



spin orbit wire

SO strength

$$H = \sum_k \{c_{k\uparrow}^{+}, c_{k\downarrow}^{+}\} \begin{pmatrix} \frac{k^2}{2m} + \alpha k & 0 \\ 0 & \frac{k^2}{2m} - \alpha k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow} \end{pmatrix}$$



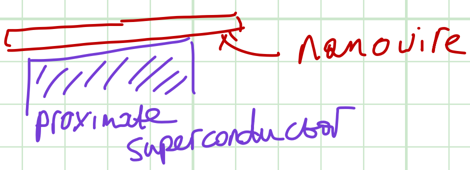
spin orbit + Zeeman wire

$\Rightarrow$ Now we can play with just the bottom band $\Rightarrow$ single species

$$H = \sum_k \{ C^\dagger_{k\uparrow}, C^\dagger_{k\downarrow} \} \begin{pmatrix} \frac{k^2}{2m} + \alpha k & B_x \\ B_x & \frac{k^2}{2m} - \alpha k \end{pmatrix} \begin{pmatrix} C_{k\uparrow} \\ C_{k\downarrow} \end{pmatrix}$$

$\Rightarrow$ In cold atoms, spin-orbit can be obtained using a Raman transition between $\uparrow$ atoms and $\downarrow$ atoms (more later)

Now add (2): We can add superconductivity using the proximity effect. The proximate superconductor does not need to be exotic.



proximity effect: tunneling of Cooper pairs from superconductor into our 1D wire

$$\mathcal{H} = \Delta C^\dagger_{k\uparrow} C^\dagger_{-k\downarrow} + \Delta^* C_{-k\downarrow} C_{k\uparrow}$$

Ok that spin directions are different they are also different for k + $-k$ in the bottom band.

We can write the complete Hamiltonian in the $\psi^\dagger = (\psi^\dagger_\uparrow \; \psi^\dagger_\uparrow \; \psi_\downarrow \; -\psi_\downarrow)$ basis. [This is the conventional choice for "historic" reasons, so we will use it]

As before we use σ Pauli matrices for the spin part and we introduce τ Pauli matrices for the particle-hole part.

$$\mathcal{H} = \psi^\dagger \left(\left[\frac{p^2}{2m} - \mu \right] \tau^z + \alpha p \sigma^z \tau^z + B \sigma^x + \Delta \tau^x \right) \psi$$

$$= \left(\psi_\uparrow^\dagger \; \psi_\downarrow^\dagger \; \psi_\downarrow \; -\psi_\uparrow\right) \left(\begin{array}{cc|cc} \frac{p^2}{2m}-\mu+\alpha p & B & \Delta & \\ B & \frac{p^2}{2m}-\mu-\alpha p & & \Delta \\ \hline \Delta & & \mu-\frac{p^2}{2m}-\alpha p & B \\ & \Delta & B & \mu-\frac{p^2}{2m}+\alpha p \end{array}\right) \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \\ \psi_\downarrow^\dagger \\ -\psi_\uparrow^\dagger \end{pmatrix}$$

$\Rightarrow$ what does Δ do to the dispersion?

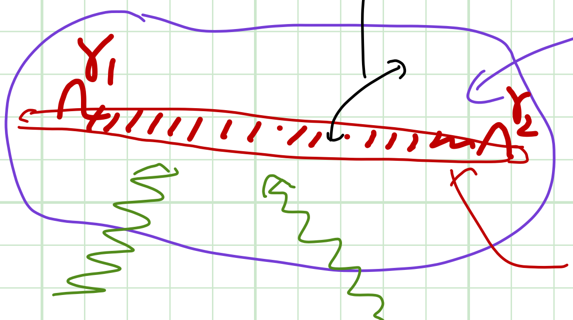
$\Rightarrow$ playing with μ, Δ, B [and $\alpha \neq 0$] let's us tune between two types of gaps $\Rightarrow$ topological and non-topo.
See artical by oreg et al for more details.

$\Rightarrow$ in real experiment one also has to control
(a) confinement in the 1D wire so that only one transverse mode fits
(b) disorder which can result in extra Majorana zero modes in the middle of the wire $\Rightarrow$ not wanted

Topological superconductivity in ultracold atoms (proposal only)

RF coupling to enable tunneling of pairs between wire and BEC

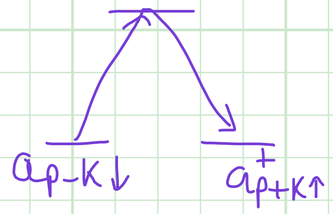
BEC cloud (proximity pairing)



1D trap $\Rightarrow$ Kitaev wire

Raman Beams
to induce spin-orbit and B-field.

Spin orbit and B-field using Raman coupling



where $2K$ = momentum transfer from the two Raman photons.

$$H_{Raman} = \begin{pmatrix} a^{+}_{p-K\downarrow} & a^{+}_{p+K\uparrow} \end{pmatrix} \begin{pmatrix} 0 & \overbrace{\frac{\Omega_1\Omega_2}{\delta_e}}^{\equiv B} \\ \frac{\Omega_1\Omega_2}{\delta_e} & 0 \end{pmatrix} \begin{pmatrix} a_{p-K\downarrow} \\ a_{p+K\uparrow} \end{pmatrix}$$

$$H_{KE} = \sum_{K\sigma} a^{+}_{p\sigma} \left(\frac{p^2}{2m} - \mu \right) a_{p\sigma}$$

Apply a spin-dependent boost:

$$\left.\begin{aligned} p-K,\downarrow &\rightarrow p\downarrow \\ p+K,\uparrow &\rightarrow p\uparrow \end{aligned}\right\}$$

$$B\, a^{+}_{p-K\downarrow}\, a_{p+K\uparrow} \rightarrow B\, a^{+}_{p\uparrow} a_{p\downarrow} \qquad a^{+}_{p\uparrow} a_{p\uparrow} \left(\frac{p^2}{2m} - \mu \right) \rightarrow a^{+}_{p\uparrow} a^{+}_{p\downarrow} \left(\frac{(p+K)^2}{2m} - \mu \right)$$

$$B\, a^{+}_{p+K\uparrow}\, a_{p-K\downarrow} \rightarrow B\, a^{+}_{p\downarrow} a_{p\uparrow} \qquad a^{+}_{p\downarrow} a_{p\downarrow} \left(\frac{p^2}{2m} - \mu \right) \rightarrow a^{+}_{p\downarrow} a_{p\downarrow} \left(\frac{(p-K)^2}{2m} - \mu \right)$$

$$H_{Raman} + H_{KE} \rightarrow \sum_p \begin{pmatrix} a^{+}_{p\uparrow} & a^{+}_{p\downarrow} \end{pmatrix} \left(\frac{p^2}{2m} + \frac{Kp}{m}\sigma^z - \mu + \frac{K^2}{2m} + \sigma^x B \right) \begin{pmatrix} a_{p\uparrow} \\ a_{p\downarrow} \end{pmatrix}$$

Topological Superconductivity in semiconducting nanowires

Lutchyn, Sau, Sarma PRL <u>105</u> 077001 (2010). } theory
Oreg, Refael, von Oppen PRL <u>105</u> 177002 (2010). }
Mourik, ..., Frolov, ..., Kouwenhoven Science <u>336</u> 1003 (2012). } experiment

Topological superconductors in ultracold atoms

Jiang et al PRL <u>106</u> 220402 (2011). ⇐ theory

Braiding Majorana fermions

See Ivanov PRL 86, 268 (2001).

Exchange of 2 Majorana fermions

$$\gamma_1 \to \gamma_2$$
$$\gamma_2 \to -\gamma_1$$
This sign has to do with crossing a branch cut

$$\Rightarrow U(\sigma_i) = \exp(\pi/4 \, \gamma_{i+1}\gamma_i) = \frac{1}{\sqrt{2}}(1+\gamma_{i+1}\gamma_i) \equiv \exp\left[i\frac{\pi}{4}(2f_i^+ f_i - 1)\right] = \exp\left[i\frac{\pi}{4}\sigma^z\right]$$

γ_1 γ_2 γ_3 γ_4 use f_1, f_3 basis

$$\gamma_1 \, \gamma_2 \, \gamma_3 \, \gamma_4 \Rightarrow e^{i\frac{\pi}{4}\sigma_1^z} \otimes I$$

$$\gamma_1 \, \gamma_2 \, \gamma_3 \, \gamma_4 \Rightarrow I \otimes e^{i\frac{\pi}{4}\sigma_3^z}$$

$$\gamma_1 \, \gamma_2 \, \gamma_3 \, \gamma_4 \Rightarrow \frac{1}{\sqrt{2}}(1+\gamma_3\gamma_2) = \frac{1}{\sqrt{2}}\left(1+i(f_3^+ + f_3)(f_1^+ - f_1)\right) = \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ This defines the Braid group set of generators

This Hamiltonian can be solved via a Bogoliubov transform

Let's begin by finding the eigenspectrum in two cases

(1) Kitaev case : $\Delta = 1$, $\mu = 0$

(2) Ordinary insulator case : $\Delta = 0$ $|\mu| > 1$

Kitaev Case :

eigenvalues:

$$\begin{vmatrix} \cos(k) - \varepsilon & -i\sin(k) \\ i\sin(k) & -\cos(k) - \varepsilon \end{vmatrix} = 0 \Rightarrow \varepsilon^2 - \cos^2(k) - \sin^2(k) = 0$$

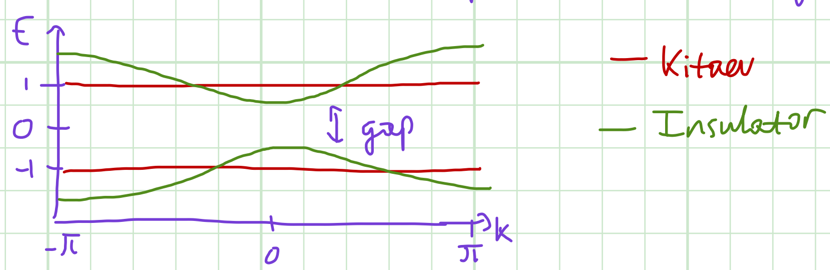
$$\varepsilon^2 = 1 \Rightarrow \varepsilon = \pm 1 \quad \text{(just as expected)}$$

Insulator Case :

eigenvalues:

$$\begin{vmatrix} \cos(k) - \mu - \varepsilon & \\ & -\cos(k) + \mu - \varepsilon \end{vmatrix} = 0 \Rightarrow \varepsilon = \pm(\cos(k) - \mu)$$

Note : Both cases correspond to a gapped spectrum :



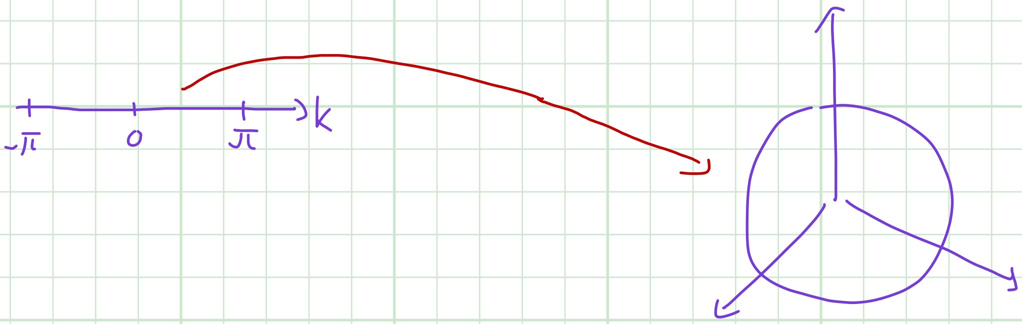
Gapped spectrum is essential for the topological argument.

$\Rightarrow$ let's rewrite the Hamiltonian in the form

$$H = \alpha(k)\,\sigma^x + \beta(k)\,\sigma^y + \gamma(k)\,\sigma^z$$

Since H is everywhere gapped, we can rescale α, β, γ so that $\alpha^2 + \beta^2 + \gamma^2 = 1$

Hence we see that the Hamiltonian is really a function from the BZ to the 3D-sphere of radius 1 (the Bloch sphere).

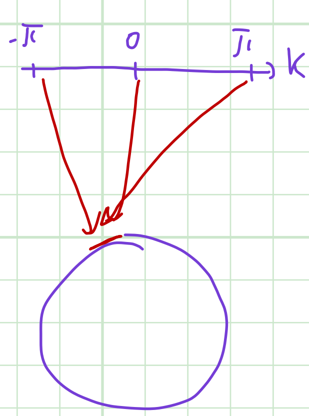


Our Hamiltonian is somewhat special: there is no σ^x, so really we have a mapping from the BZ onto the unit Ring.

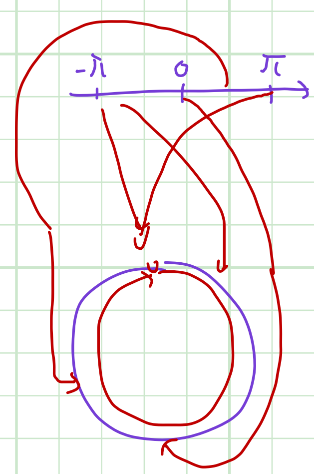
moreover $H(-\pi) = H(\pi)$ so the mapping should be periodic!

What kind of mappings are allowed:

Trivial map (insulator)



Kitaev model



To go from trivial map to non-trivial one we must

(a) close the gap (at some k-point $\Rightarrow$ this let's us add + subtract loops)

(b) deform H into σ^x direction

So as long as (a) + (b) do not occur we have topological protection.

Topology in eigen functions.

Let us now look at the eigen functions of H:

These are the Bogoliubov operators $\gamma_k^\dagger = u_k c_k + v_k c_{-k}^\dagger$.

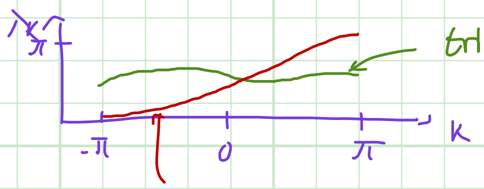
What can we say about u_k and v_k?

(1) since H is real, we can choose u_k and v_k to be real as well

(2) since $u_k^2 + v_k^2 = 1$, we can introduce an angle λ_k, $u_k = \sin(\lambda_k)$
$v_k = \cos(\lambda_k)$

(3) Eigenvalues are determined up to a phase: if $\{u_k, v_k\}$ is an eigenvalue so is $\{-u_k, -v_k\}$.

Q: what does λ_k do as k goes around the BZ?



trivial case $\lambda_{-\pi} = \lambda_\pi$ $\Rightarrow$ trivial insulator case

non-trivial case $\lambda_{-\pi} = \lambda_\pi \pm \pi$ $\Rightarrow$ Kitaev phase

What is non-Abelian statistics $\Rightarrow$ next time?